Malaysian Journal of Mathematical Sciences 15(3): 489–507 (2021)

Malaysian Journal of Mathematical Sciences

Journal homepage: https://einspem.upm.edu.my/journal

A New Approach to Numerical Solution of Nonlinear Partial Mixed Volterra-Fredholm Integral Equations via Two-Dimensional Triangular Functions

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Received: 4 March 2020
Accepted: 27 April 2021

Abstract

This paper proposes a numerical procedure for solving the nonlinear partial mixed Volterra-Fredholm integro-differential equations by two-dimensional triangular function (2D-TFs). The integration and differentiation in two-dimensional spaces have been presented for an operational matrix on triangular functions, whereas by converting the nonlinear partial mixed Volterra-Fredholm integro-differential equation to a system of algebraic by using these matrices. Some numerical examples, have been proposed to obtain the accuracy and effectiveness of the method.

Keywords: nonlinear equations; partial mixed Volterra-Fredholm integral equations; operational matrix; two-dimensional triangular functions.

1 Introduction

The theory and applications of differential equations contain the majority of scope in applied science and technology. The partial and integro-differential equations arise in wide fields of applied problems such as mechanics, physics, engineering, astrophysics, and even in the biology. For example M. I. Berenguer et. al have applied computational method in [5], the solution of special class of physical equation has been given in [16], the concept of integral equations has been presented in [10] and recently in [7] has presented some new method. The concept of two-dimensional integral equation appears frequently in the theory of plasma physics and finding the solution of these two-dimensional integral equations is usually necessary [9]. There is no numerical approach with high precision to find solutions to such equations. In the following, we introduce the special class of two-dimensional integral equation entitled "nonlinear partial mixed Volterra-Fredholm integral equation" and we construct the numerical solution based on an operational matrix.
The nonlinear partial mixed Volterra-Fredholm integral equation is defined as follows:

$$f\left(s, t, u, u_s, u_t, \ldots, \frac{\partial^{n+m} u(s,t)}{\partial s^n \partial t^m}\right) + \lambda \int_0^s \int_\Omega k(s,t,x,y,u(x,y))dydx = 0,$$

with known extra conditions, as well as the function $u(s,t)$ is an unknown which must be determined, the analytical functions $f(s,t)$ and $k(s,t,x,y)$ defined on $D = [0,T] \times \Omega$ and $D \times \Omega^2$, respectively; Ω is a closed bounded region in R^n $(n = 1,2,3)$ with piecewise smooth boundary $\partial\Omega$ [6]. Nowadays, some numerical methods and attempts have been made to find the numerical solution of the mixed Volterra-Fredholm for instance: He's variational iteration method [19] and Homotopy perturbation method [18]. The TFs method used properly to approximate the solution of Fredholm and Volterra integral equations of the second kind [3, 2]. Maleknejad in [12] have applied a TFs method to find the solution of NVFIs. In [15] the Bernstein polynomials also developed to solve the two-dimensional integral equation. Maleknejad and Mahdiani have attempted to applied 2D-BPFs to evaluated the numerical approximation of the nonlinear mixed Volterra-Fredholm integral equations [14], and Maleknejad in [13] have applied such operational matrix and TFs for an especial class of mixed Volterra-Fredholm integral equations. The authors have applied operational matrix for solving two-dimensional (2D) nonlinear integro-differential equations by BPFs [1]. Ebadian, and Khajehnasiri has used an operational matrix to find the solution of the nonlinear Volterra integro-differential [11]. Recently, J. Xie in [17] has applied Block-Pulse functions for another class of nonlinear equations known as Volterra-Fredholm-Hammerstein integral equations. In current work, the 2D-TFs have been used to evaluate the proper solution of the Eq. (1).

This article is organized as follows. In Section 2, the TFs theory and their properties have been presented. In Section 3, we introduce the method's application. For better illustration the accuracy of the proposed methods we displayed some numerical results in Section 4. Finally, at the end of the paper, we were given some remarks in Section 5.

2 A Brief Presentation and Properties of the Triangular Functions

Throughout this section, we will provide a glimpse into the properties of triangular functions.

2.1 1-D Triangular Functions

We usually define the Triangular Functions, which known as one-dimensional triangular functions (1D-TFs) with an m-set on interval [0,1), where ith left hand as well as ith right-hand functions of which are presented such as:

$$T_i^1(s) = \begin{cases} 1 - \frac{s-ih}{h}, & ih \le s < (i+1)h, \\ 0, & \text{otherwise}, \end{cases}$$

$$T_i^2(s) = \begin{cases} \frac{s-ih}{h}, & ih \le s < (i+1)h, \\ 0, & \text{otherwise}, \end{cases}$$

where $i = 0, ..., m-1, h = \frac{1}{m}$. Someone could show that

$$T_i^1(s) + T_i^2(s) = \Phi_i(s), \tag{1}$$

where $\Phi_i(s)$ is the block-pulse function in ith point would be represent as

$$\Phi_i(s) = \begin{cases} 1, & ih \le s < (i+1)h, \\ 0, & \text{otherwise}. \end{cases}$$

Clearly $\{T_i^1(s)\}_{i=0}^{m-1}$ and $\{T_i^2(s)\}_{i=0}^{m-1}$ are disjointed. So we have

$$\begin{aligned} T1(s) \cdot T1^T(s) &\simeq diag(T1(s)) = \tilde{T}1(s), \\ T1(s) \cdot T2^T(s) &\simeq 0_{m\times m}, \\ T2(s) \cdot T1^T(s) &\simeq 0_{m\times m}, \\ T2(s) \cdot T2^T(s) &\simeq diag(T2(s)) = \tilde{T}1(s), \end{aligned} \tag{2}$$

where $\tilde{T}1(s)$ and $\tilde{T}2(s)$ are represented as $m \times m$ diagonal matrices [4]. To find the orthogonality for 1D-TFs we can see [8], which is,

$$\int_0^1 T_i^p(s) T_j^q(s) = \Delta_{p,q} \delta_{i,j},$$

such that δ_{ij} denotes the Kronecker delta function and

$$\Delta_{p,q} = \begin{cases} \frac{h}{3}, & p = q \in \{1,2\}, \\ \frac{h}{6}, & p \neq q. \end{cases}$$

If we reconsider and define

$$T_1(s) = [T_1^0(s), T_1^1(s), \cdots, T_1^{m-1}(s)]^T,$$

$$T_2(s) = [T_2^0(s), T_2^1(s), \cdots, T_2^{m-1}(s)]^T,$$

and

$$T(s) = \begin{bmatrix} T_1(s) \\ T_2(s) \end{bmatrix}.$$

So,

$$\begin{aligned}\int_0^1 T1(t)T1^T(t)dt &= \int_0^1 T2(t)T2^T(t)dt = \frac{h}{3}I_{m\times m},\\ \int_0^1 T1(t)T2^T(t)dt &= \int_0^1 T2(t)T1^T(t)dt = \frac{h}{6}I_{m\times m}.\end{aligned}$$

Expressing $\int_0^s T1(\tau)d\tau$ and $\int_0^s T2(\tau)d\tau$ in which terms including of 1D-TFs follows

$$\int_0^s T1(\tau)d\tau = P1s\cdot T1(s) + P2s\cdot T2(s), \tag{3}$$

$$\int_0^s T1(\tau)d\tau = P1s\cdot T1(s) + P2s\cdot T2(s), \tag{4}$$

where

$$P1s = \frac{h}{2}\begin{pmatrix} 0 & 1 & 1 & \dots & 1\\ 0 & 0 & 1 & \dots & 1\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & 0\end{pmatrix},$$

$$P2s = \frac{h}{2}\begin{pmatrix} 1 & 1 & 1 & \dots & 1\\ 0 & 1 & 1 & \dots & 1\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & 1\end{pmatrix}.$$

Then,

$$\int_0^s T(\tau)d\tau \simeq P.T(s),$$

such that matrix P, which is the operational matrix of integration could be shown in the 1D-TF domain such as:

$$P = \begin{bmatrix} P1s & P2s\\ P1s & P2s\end{bmatrix}.$$

So, the approximation of integration for function $f(\tau)$ can be estimated by

$$\int_0^s f(\tau)d\tau \simeq \int_0^s C^T\cdot T(\tau)d\tau \simeq C^T\cdot P\cdot T(s).$$

To find more details about 1D-TFs see [8].

2.2 Two-Dimensional Triangular Functions

We define 2D-TFs as a set of $m_1 \times m_2$ on the square block of $([0,1) \times [0,1))$ by

$$T^{1,1}_{i,j}(s,t) = \begin{cases} \left(1 - \frac{s-ih_1}{h_1}\right)\left(1 - \frac{t-jh_2}{h_2}\right) & ih_1 \leqslant s < (i+1)h_1, \\ & jh_2 \leqslant t < (j+1)h_2, \\ 0 & \text{otherwise}, \end{cases} \tag{5}$$

$$T^{1,2}_{i,j}(s,t) = \begin{cases} \left(1 - \frac{s-ih_1}{h_1}\right)\left(\frac{t-jh_2}{h_2}\right) & ih_1 \leqslant s < (i+1)h_1, \\ & jh_2 \leqslant t < (j+1)h_2, \\ 0 & \text{otherwise}, \end{cases} \tag{6}$$

$$T^{2,1}_{i,j}(s,t) = \begin{cases} \left(\frac{s-ih_1}{h_1}\right)\left(1 - \frac{t-jh_2}{h_2}\right) & ih_1 \leqslant s < (i+1)h_1, \\ & jh_2 \leqslant t < (j+1)h_2, \\ 0 & \text{otherwise}, \end{cases} \tag{7}$$

$$T^{2,2}_{i,j}(s,t) = \begin{cases} \left(\frac{s-ih_1}{h_1}\right)\left(\frac{t-jh_2}{h_2}\right) & ih_1 \leqslant s < (i+1)h_1, \\ & jh_2 \leqslant t < (j+1)h_2, \\ 0 & \text{otherwise}, \end{cases} \tag{8}$$

where $i = 0,1,2,\cdots,m_1-1$, $j = 0,1,2,\cdots,m_2-1$ and $h_1 = \frac{1}{m_1}$, $h_2 = \frac{1}{m_2}$, m_1 and m_2 are positive integers as well as arbitrary.

Clearly we have,

$$\begin{aligned} T^{1,1}_{i,j}(s,t) &= T^1_i(s)\cdot T^1_j(t), \\ T^{1,2}_{i,j}(s,t) &= T^1_i(s)\cdot T^2_j(t), \\ T^{2,1}_{i,j}(s,t) &= T^2_i(s)\cdot T^1_j(t), \\ T^{2,2}_{i,j}(s,t) &= T^2_i(s)\cdot T^2_j(t). \end{aligned} \tag{9}$$

Furthermore,

$$T^{1,1}_{i,j}(s,t) + T^{1,2}_{i,j}(s,t) + T^{2,1}_{i,j}(s,t) + T^{2,2}_{i,j}(s,t) = \phi_{i,j}(s,t),$$

where $\phi_{i,j}(s,t)$ is the (i,j)th component of block-pulse function which is defined on $ih_1 \leqslant s < (i+1)h_1$ and $jh_2 \leqslant t < (j+1)h_2$ as

$$\phi_{i,j}(s,t) = \begin{cases} 1 & ih_1 \leqslant s < (i+1)h_1, \\ & jh_2 \leqslant t < (j+1)h_2, \\ 0 & \text{otherwise}. \end{cases} \tag{10}$$

Similar to the 1D subsection the important part of the properties of 2D-TFs, are disjointness and orthogonality. For every set of $\{T^{11}_{ij}(s,t)\}$, $\{T^{12}_{ij}(s,t)\}$, $\{T^{21}_{ij}(s,t)\}$ and $\{T^{22}_{ij}(s,t)\}$ are clearly:

1. **Disjointness**

   The two-dimensional triangular functions are disjoined with each other, i.e.

$$T^{p_1,q_1}_{i_1,j_1}(s,t)\cdot T^{p_2,q_2}_{i_2,j_2}(s,t)\simeq\left\{\begin{array}{ll} T^{p_1,q_1}_{i_1,j_1}(s,t) & p_1=p_2,\quad q_1=q_2,\\ & i_1=i_2,\quad j_1=j_2,\\ 0 & \quad\text{otherwise},\end{array}\right\},$$

   for $p,q\in\{1,2\}, i_1,i_2=0,1,2,\cdots,m_1-1$, and $j_1,j_2=0,1,2,\cdots,m_2-1$.

2. **Orthogonality**

   We can prove that the 2D-TFs component has the property of orthogonality with the others, i.e.

$$\int_0^1\int_0^1 T^{p_1,q_1}_{i_1,j_1}(s,t)\cdot T^{p_2,q_2}_{i_2,j_2}(s,t)dsdt=\Delta_{p_1,p_2}\delta_{i_1,i_2}\cdot\Delta_{q_1,q_2}\delta_{j_1,j_2},$$

   where δ denotes the Kronecker delta function, and

$$\Delta_{\alpha,\beta}=\left\{\begin{array}{ll}\frac{h}{3}, & \alpha=\beta\in\{1,2\},\\ \frac{h}{6}, & \alpha\neq\beta.\end{array}\right. \tag{11}$$

Nevertheless, if

$$\begin{aligned}
T_{11}(s,t)&=[T^{1,1}_{0,0}(s,t),T^{1,1}_{0,1}(s,t),\ldots,T^{1,1}_{m_1-1,m_2-1}(s,t)]^T,\\
T_{12}(s,t)&=[T^{1,2}_{0,0}(s,t),T^{1,2}_{0,1}(s,t),\ldots,T^{1,2}_{m_1-1,m_2-1}(s,t)]^T,\\
T_{21}(s,t)&=[T^{2,1}_{0,0}(s,t),T^{2,1}_{0,1}(s,t),\ldots,T^{2,1}_{m_1-1,m_2-1}(s,t)]^T,\\
T_{22}(s,t)&=[T^{2,2}_{0,0}(s,t),T^{2,2}_{0,1}(s,t),\ldots,T^{2,2}_{m_1-1,m_2-1}(s,t)]^T,
\end{aligned}$$

then, we can defined the 2D-TF vector of $T(s,t)$, as

$$T(s,t)=\begin{bmatrix}T_{11}(s,t)\\ T_{12}(s,t)\\ T_{21}(s,t)\\ T_{22}(s,t)\end{bmatrix}_{4m_1m_2\times 1}. \tag{12}$$

By eliminating the pair variables (s,t) from $T(s,t), T_{11}(s,t), T_{12}(s,t), T_{21}(s,t)$ and $T_{22}(s,t)$, we simplify the concept and understanding. In light of previous representations, it follows that

$$\begin{aligned}
T_{11}\cdot T_{11}^T&\simeq\begin{bmatrix}T^{1,1}_{11} & 0 & \cdots & 0\\ 0 & T_{11}^T & \cdots & \vdots\\ \vdots & \vdots & \ddots & 0\\ 0 & 0 & \ldots & T^{1,1}_{m_1-1,m_2-1}\end{bmatrix}=diag(T_{1,1}),\\
T_{11}\cdot T_{12}^T&\simeq 0_{m_1m_2\times m_1m_2},\\
T_{11}\cdot T_{21}^T&\simeq 0_{m_1m_2\times m_1m_2},\\
T_{11}\cdot T_{22}^T&\simeq 0_{m_1m_2\times m_1m_2}.
\end{aligned}$$

We can see similarly that above relations are completely satisfied for $T_{12}(s,t)$, $T_{21}(s,t)$ and $T_{22}(s,t)$. Therefore,

$$T \cdot T^T \simeq \begin{bmatrix} diag(T_{1,1}) & 0_{m_1m_2\times m_1m_2} & 0_{m_1m_2\times m_1m_2} & 0_{m_1m_2\times m_1m_2} \\ 0_{m_1m_2\times m_1m_2} & diag(T_{1,2}) & 0_{m_1m_2\times m_1m_2} & 0_{m_1m_2\times m_1m_2} \\ 0_{m_1m_2\times m_1m_2} & 0_{m_1m_2\times m_1m_2} & diag(T_{2,1}) & 0_{m_1m_2\times m_1m_2} \\ 0_{m_1m_2\times m_1m_2} & 0_{m_1m_2\times m_1m_2} & 0_{m_1m_2\times m_1m_2} & diag(T_{2,2}) \end{bmatrix}, \qquad (13)$$

or

$$T(s,t) \cdot T^T(s,t) \simeq diag(T(s,t)) = \tilde{T}(s,t).$$

Also,

$$T(s,t) \cdot T^T(s,t) \cdot X \simeq \tilde{X} \cdot T(s,t),$$

in which $\tilde{X} = diag(X)$ and X represented as $4m_1m_2$-vector.

By the disjointness property of $T_{11}(s,t)$, $T_{12}(s,t)$, $T_{21}(s,t)$ and $T_{22}(s,t)$also implies that for every $(4m_1m_2 \times 4m_1m_2)$-matrix A

$$T^T(s,t) \cdot A \cdot T(s,t) \simeq \widehat{A} \cdot T(s,t),$$

where $\hat{A}$ is a $4m_1m_2$-vector with elements that are the same diagonal matrix A.

2.3 2D-TFs Expansion

We may extended using 2D-TFs and define the function $u(s,t)$ on the square block of $([0,1) \times [0,1))$ as follow.

$$\begin{aligned} u(s,t) &\simeq \sum_{i=0}^{m_1-1}\sum_{j=0}^{m_2-1} c_{i,j}T_{i,j}^{1,1} + \sum_{i=0}^{m_1-1}\sum_{j=0}^{m_2-1} d_{i,j}T_{i,j}^{1,2} + \sum_{i=0}^{m_1-1}\sum_{j=0}^{m_2-1} e_{i,j}T_{i,j}^{1,1} + \sum_{i=0}^{m_1-1}\sum_{j=0}^{m_2-1} l_{i,j}T_{i,j}^{2,2}, \\ &= C_1^T \cdot T_{11}(s,t) + C_2^T \cdot T_{12}(s,t) + C_3^T T_{21}(s,t) + C_4^T \cdot T_{22}(s,t), \\ &= C^T \cdot T(s,t), \end{aligned} \qquad (14)$$

such that C is a $4m_1m_2$ -vector defined by

$$C = [C_1^T \; C_2^T \; C_3^T \; C_4^T]^T, \qquad (15)$$

and $T(s,t)$ is given in Eq. (12). The 2D-TFs components in $C1, C2, C3,$ and $C4$ could be evaluated by the function $u(s,t)$ at grid points s_i and t_j where $s_i = ih_1$ and $t_j = jh_2$, for different i and j. So,

$$\begin{aligned} C_{1k} &= c_{i,j} = u(s_i,t_j), \quad C_{2k} = d_{i,j} = u(s_i,t_{j+1}), \\ C_{3k} &= e_{i,j} = u(s_{i+1},t_j), \quad C_{14k} = l_{i,j} = u(s_{i+1},t_{j+1}), \end{aligned} \qquad (16)$$

where $k = im_2 + j$ and $i = 0, 1, \cdots, m_1 - 1$, $j = 0, 1, \cdots, m_2 - 1$, the $4m_1m_2$-vector C is called the 2D-TF coefficient vector. Furthermore, we can approximated using 2D-TFs for the positive integer powers of a function $u(s,t)$ as

$$[u(s,t)]^p \simeq C_p^T \cdot T(s,t),$$

such that C_p is a column vector such that those elements are p^{th} powers of the elements for the vector C.

Suppose that $k(s,t,x,y)$ be a four variables function on $([0,1) \times [0,1) \times [0,1) \times [0,1))$. We can approximated with respect to 2D-TFs such as:

$$k(s,t,x,y) = T^T(s,t) \cdot K \cdot T(x,y), \tag{17}$$

where both of $T(s,t)$ and $T(x,y)$ are dimensional vectors of 2D-TFs $4m_1m_2$ and $4m_3m_4$, respectively, K is the 2D-TF coefficients matrix of $(4m_1m_2) \times (4m_3m_4)$ and may be computed by sampling the function $k(s,t,x,y)$ [2, 1]. Now, we using (14), every element of 1D-TFs which is a m_1-set, can be expanded with respect to an $m_1 \times m_2$-set of 2D-TFs. For $T_0^1(s)$ we have

$$T_0^1(s) \simeq [C1^T \quad C2^T \quad C3^T \quad C4^T] \cdot T(s,t).$$

Since t does not appear in $T_0^1(s)$, Eqs. (16) lead to

$$\begin{aligned} C1_k &= C2_k = T_0^1(s_i) = \begin{cases} 1 & k = 1, 2, \cdots, m_2, \\ 0 & k = m_2 + 1, \cdots, m_1m_2, \end{cases} \\ C3_k &= C4_k = T_0^1(s_{i+1}) = 0, \quad k = 1, 2, \cdots, m_2. \end{aligned}$$

Another components of $T(s)$ could be expanded in such way. therefore

$$T(s) \simeq M_{2m_1 \times 4m_1m_2} \cdot T(s,t),$$

where

$$M = \begin{bmatrix} I_{m_1 \times m_1} & I_{m_1 \times m_1} & 0_{m_1 \times m_1} & 0_{m_1 \times m_1} \\ 0_{m_1 \times m_1} & 0_{m_1 \times m_1} & I_{m_1 \times m_1} & I_{m_1 \times m_1} \end{bmatrix} \otimes 1_{1 \times m_2},$$

and $\otimes$ denotes the Kronecker product which can represent for two matrices P and Q as

$$P \otimes Q = (p_{i,j}Q). \tag{18}$$

Hence, the Kronecker product of vector $1_{1 \times m_2}$ from right side in any matrix or vector produces m_2-repetition of each component in the row in which it appears.

2.4 The Operational Matrix of TFs for Integration

Within this subsection, we present the TFs for Integration that are derived from the proposed operational matrix. In this approach we try to evaluate the double integral of $\int_0^s \int_0^1 u(x,y)dydx$.

We need to evaluate above integral for vector $T(x, y)$. From Eqs. (9), we could evaluate separately the integrals with respect to x and y. During this approach to find the evaluation, we represent an operational matrix for approximating the integration with respect to x in $\int_0^s \int_0^1 T(x, y)dydx$. Let $Ts(x)$ be the x-components of $T(x, y)$. We may write

$$T_s(x) = \left(\begin{array}{c} \left.\begin{array}{c} T_0^1(x) \\ \vdots \\ T_0^1(x) \end{array}\right\} 2m_2\ times \\ \left.\begin{array}{c} T_1^1(x) \\ \vdots \\ T_1^1(x) \end{array}\right\} 2m_2\ times \\ \vdots \\ \left.\begin{array}{c} T_{m_1-1}^2(x) \\ \vdots \\ Tm_1 - 1^2(x) \end{array}\right\} 2m_2\ times \end{array} \right)_{4m_1m_2\times 1} = \begin{bmatrix} T_1(x) \\ T_1(x) \\ T_2(x) \\ T_2(x) \end{bmatrix}_{4m_1\times 1} \otimes 1_{m_2\times 1}$$

Using Eqs. (3) and (4), we can approximate the integral of $Ts(x)$ as

$$\begin{aligned} \int_0^s T_s(x)dx &\simeq \left(\begin{bmatrix} P1_s & P2_s \\ P1_s & P2_s \\ P1_s & P2_s \\ P1_s & P2_s \end{bmatrix} \otimes 1_{m_2\times 1} \right) \cdot T(s), \\ &\simeq \left(\begin{bmatrix} P1_s & P2_s \\ P1_s & P2_s \\ P1_s & P2_s \\ P1_s & P2_s \end{bmatrix} \otimes 1_{m_2\times 1} \right) \cdot M \cdot T(s,t), \\ &= E \cdot T(s,t), \end{aligned} \tag{19}$$

in which E is a $(4m_1m_2 \times 4m_1m_2)$-matrix as follows:

$$\begin{aligned} E &= \left(\begin{bmatrix} P1_s & P2_s \\ P1_s & P2_s \\ P1_s & P2_s \\ P1_s & P2_s \end{bmatrix} \otimes 1_{m_2\times 1} \right) \cdot \left(\begin{bmatrix} I & I & 0 & 0 \\ 0 & 0 & I & I \end{bmatrix}_{2m_1\times 4m_1} \otimes 1_{1\times m_2} \right), \\ &= \begin{bmatrix} P1_s & P1_s & P2_s & P2_s \\ P1_s & P1_s & P2_s & P2_s \\ P1_s & P1_s & P2_s & P2_s \\ P1_s & P1_s & P2_s & P2_s \end{bmatrix} \otimes 1_{m_2\times 1}. \end{aligned} \tag{20}$$

More details may be found in [13].

2.4.1 Product Properties

In this subsection, we estimates the product and their properties of which for the problem. We need the following procedure to approximate $T(s, t) \cdot T^T(s, t)$.

Each component of this matrix from Eqs. (9), can be represented separately as the production of two terms with respect to variables s and t. Hence,

$$T(s,t)\cdot T^T(s,t) = [A_{ij}(s)B_{ij}(t)]_{i,j=1,2,\cdots,4m_1m_2}, \tag{21}$$

such that $A(s)$ and $B(t)$ are $(4m_1m_2 \times 4m_1m_2)$-matrices and must be computed. The disjoint property of $\{T_i^1(s)\}_{i=0}^{m_1-1}$ and $\{T_i^2(s)\}_{i=0}^{m_1-1}$ Eqs. (2) and (9) that

$$\begin{aligned} A(s) &= \left(\begin{bmatrix} T1_{(s)} \\ T1_{(s)} \\ T2_{(s)} \\ T2_{(s)} \end{bmatrix} \otimes 1_{m_2\times 1}\right)\cdot\left(\begin{bmatrix} T1^T(s), & T1^T(s), & T2^T(s), & T2^T(s) \end{bmatrix}\otimes 1_{1\times m_2}\right), \\ &\simeq \begin{bmatrix} \widetilde{T1}(s) & \widetilde{T1}(s) & 0 & 0 \\ \widetilde{T1}(s) & \widetilde{T1}(s) & 0 & 0 \\ 0 & 0 & \widetilde{T2}(s) & \widetilde{T2}(s) \\ 0 & 0 & \widetilde{T2}(s) & \widetilde{T2}(s) \end{bmatrix} \otimes 1_{m_2\times m_2}. \end{aligned} \tag{22}$$

Furthermore, the disjointness property of $\{T_j^1(t)\}_{j=0}^{m_2-1}$ and $\{T_j^2(t)\}_{j=0}^{m_2-1}$ implies that

$$B(t) = \begin{bmatrix} 1_{m_1\times 1}\otimes T1 \\ 1_{m_1\times 1}\otimes T2 \\ 1_{m_1\times 1}\otimes T1 \\ 1_{m_1\times 1}\otimes T2 \end{bmatrix}\cdot\begin{bmatrix} 1_{1\times m_1}\otimes T1, & 1_{1\times m_1}\otimes T2, & 1_{1\times m_1}\otimes T1, & 1_{1\times m_1}\otimes T2 \end{bmatrix},$$

and in definition the (t) term in $T1(t)$ and $T2(t)$ is omitted, for reliability. So,

$$B(t) \simeq \begin{bmatrix} B_q(t) & B_q(t) \\ B_q(t) & B_q(t) \end{bmatrix}, \tag{23}$$

where $B_q(t)$ is a $(2m_1m_2 \times 2m_1m_2)$-matrix of the form

$$B_q(t) \simeq \begin{bmatrix} 1_{m_1\times m_1}\otimes(\tilde{T1}\cdot\tilde{T1}) & 1_{m_1\times m_1}\otimes(\tilde{T1}\cdot\tilde{T2}) \\ 1_{m_1\times m_1}\otimes(\tilde{T1}\cdot\tilde{T2}) & 1_{m_1\times m_1}\otimes(\tilde{T2}\cdot\tilde{T2}) \end{bmatrix}.$$

We have to reform how the matrices $A(s)$ and $B(t)$ are approximated in Eqs. (22) and (23). In spite of the fact that all blocks in $B(t)$ are diagonal matrices, one may rewrite the Eq. (21) such as

$$T(s,t)\cdot T^T(s,t) \simeq \begin{bmatrix} \widetilde{Q11}(s,t) & \widetilde{Q12}(s,t) & 0 & 0 \\ \widetilde{Q12}(s,t) & \widetilde{Q13}(s,t) & 0 & 0 \\ 0 & 0 & \widetilde{Q21}(s,t) & \widetilde{Q22}(s,t) \\ 0 & 0 & \widetilde{Q22}(s,t) & \widetilde{Q23}(s,t) \end{bmatrix}, \tag{24}$$

in which by considering

$$\begin{aligned} A1(s) &= trace(\widetilde{T1}(s) \otimes 1_{m_2 \times m_2}), \\ A2(s) &= trace(\widetilde{T2}(s) \otimes 1_{m_2 \times m_2}), \\ B1(t) &= trace(1_{m_1 \times m_1} \otimes (\widetilde{T1}(t) \cdot \widetilde{T1}(t))), \\ B2(t) &= trace(1_{m_1 \times m_1} \otimes (\widetilde{T1}(t) \cdot \widetilde{T2}(t))), \\ B3(t) &= trace(1_{m_1 \times m_1} \otimes (\widetilde{T2}(t) \cdot \widetilde{T2}(t))), \end{aligned}$$

we have,

$$\begin{aligned} (Q11)_k &= (A1)_k \cdot (B1)_k, \\ (Q12)_k &= (A1)_k \cdot (B2)_k, \\ (Q13)_k &= (A1)_k \cdot (B3)_k, \\ (Q21)_k &= (A2)_k \cdot (B1)_k, \\ (Q22)_k &= (A2)_k \cdot (B2)_k, \\ (Q23)_k &= (A2)_k \cdot (B3)_k, \end{aligned} \tag{25}$$

for $k = 1, 2, \cdots, m_1 m_2$. Now, let X be a $4m1m2$-vector as

$$X = [X_1^T, \quad X_2^T, \quad X_3^T, \quad X_4^T]^T, \tag{26}$$

where $X1$, $X2$, $X3$ and $X4$ are $m_1 m_2$-vectors. It can be derived from Eqs. (24) to (25) that

$$\begin{aligned} T(s,t) \cdot T^T(s,t) \cdot X &\simeq \begin{bmatrix} \widetilde{Q11}(s,t) & \widetilde{Q12}(s,t) & 0 & 0 \\ \widetilde{Q12}(s,t) & \widetilde{Q13}(s,t) & 0 & 0 \\ 0 & 0 & \widetilde{Q21}(s,t) & \widetilde{Q22}(s,t) \\ 0 & 0 & \widetilde{Q22}(s,t) & \widetilde{Q23}(s,t) \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}, \\ &= \begin{bmatrix} \widetilde{X1} \cdot \widetilde{B1} & \widetilde{X2} \cdot \widetilde{B2} & 0 & 0 \\ \widetilde{X1} \cdot \widetilde{B2} & \widetilde{X2} \cdot \widetilde{B3} & 0 & 0 \\ 0 & 0 & \widetilde{X3} \cdot \widetilde{B1} & \widetilde{X4} \cdot \widetilde{B2} \\ 0 & 0 & \widetilde{X3} \cdot \widetilde{B2} & \widetilde{X4} \cdot \widetilde{B2} \end{bmatrix} \begin{bmatrix} A_1 \\ A_1 \\ A_2 \\ A_2 \end{bmatrix}, \\ &= \begin{bmatrix} \widetilde{B1} & \widetilde{B2} & 0 & 0 \\ \widetilde{B2} & \widetilde{B3} & 0 & 0 \\ 0 & 0 & \widetilde{B1} & \widetilde{B2} \\ 0 & 0 & \widetilde{B2} & \widetilde{B3} \end{bmatrix} \begin{bmatrix} \widetilde{X1} & 0 & 0 & 0 \\ 0 & \widetilde{X2} & 0 & 0 \\ 0 & 0 & \widetilde{X3} & 0 \\ 0 & 0 & 0 & \widetilde{X4} \end{bmatrix} \begin{bmatrix} A_1 \\ A_1 \\ A_2 \\ A_2 \end{bmatrix}, \\ &= R(t) \cdot \tilde{X} \cdot Ts(s). \end{aligned} \tag{27}$$

Hence, components s and t can be decomposed into vectors $Ts(s)$ and matrix $R(t)$, respectively. In addition, that s-components are aggregated in the last vector. For further details, one may refer [13].

2.5 Operational Matrix of TFs for Differentiation

Finding an approximate solution involves constructing a differentiation for the problem by adding its conditions. This subsection presents a differentiation matrix based on the proposed operational matrix. Using these initial conditions based on proposed operational matrices, we can write :

$$\begin{aligned}
u(s,t) &= U^T T(s,t), \\
u(s,0) &= U_{s0}^T T(s,t), \\
u(0,t) &= U_{0t}^T T(s,t), \\
u_t(s,t) &= U_t^T T(s,t), \\
u_s(s,t) &= U_s^T T(s,t), \\
u_t(s,0) &= U_{ts0}^T T(s,t), \\
u_{tt}(s,t) &= U_{tt}^T T(s,t), \\
u_s(0,s) &= U_{s0t}^T T(s,t), \\
u_{ss}(s,t) &= U_{ss}^T T(s,t), \\
u_{st}(s,t) &= U_{st}^T T(s,t),
\end{aligned} \tag{28}$$

where some boundary and initial conditions simply can be calculated. We can use the following procedure to operational matrix of TFs for differentiation for U_t^T, U_s^T, U_{tt}^T, $U_{ss}T$, and U_{st}^T. To find operational matrix of TFs for U_t^T we can write the fundamental theorem of calculus down as follows:

$$u(s,t) - u(s,0) = \int_0^t u_t(s,\tau)d\tau. \tag{29}$$

From (28), we obtain

$$\begin{aligned}
U^T T(s,t) - U_{s0}^T T(s,t) &= \int_0^t U_t^T T(s,\tau)d\tau, \\
&= U_t^T \int_0^t T(s,\tau)d\tau = U_t^T P T(s,t).
\end{aligned}$$

Thus, we get

$$U^T - U_{s0}^T = U_t^T P. \tag{30}$$

Then,

$$U_t^T = (U^T - U_{s0}^T)P^{-1}. \tag{31}$$

In similar way, to find operational matrix of TFs for U_s^T we have

$$U_s^T = (U^T - U_{0t}^T)P^{-1}. \tag{32}$$

In the similar approach, the fundamental theorem of calculus can be used to approximate the operational matrix of TFs for U_{tt}^T as follows;

$$u_t(s,t) - u_t(s,0) = \int_0^t u_{tt}(s,\tau)d\tau. \tag{33}$$

By using (28), we have

$$\begin{aligned} U_t^T T(s,t) - U_{ts0}^T T(s,t) &= \int_0^t U_{tt}^T T(s,\tau)d\tau, \\ &= U_{tt}^T \int_0^t T(s,\tau)d\tau = U_{tt}^T P T(s,t). \end{aligned}$$

Thus, we get

$$U_t^T - U_{ts0}^T = U_{tt}^T P. \tag{34}$$

Then,

$$U_{tt}^T = (U_t^T - U_{ts0}^T)P^{-1}. \tag{35}$$

In the same approach, we can approximate $u_{ss}(s,t)$ in the form of the following equation

$$U_{ss}^T = (U_s^T - U_{s0t}^T)P^{-1}. \tag{36}$$

At the end of our procedure we can approximate $u_{st}(s,t)$ and use the following procedure:

$$u_t(s,t) - u_t(t,0) = \int_0^t u_{st}(t,\tau)d\tau. \tag{37}$$

Hence,

$$\begin{aligned} U_s^T T(s,t) - U_{s0t}^T T(s,t) &= \int_0^t U_{st}^T T(s,\tau)d\tau, \\ &= U_{st}^T \int_0^t T(s,\tau)d\tau = U_{st}^T P T(s,t). \end{aligned}$$

So, we get

$$U_s^T - U_{s0t}^T = U_{st}^T P. \tag{38}$$

Then, we have

$$U_{st}^T = (U_s^T - U_{s0t}^T)P^{-1}. \tag{39}$$

Finally, we could using such proposed method to evaluate and expand the operational matrix for order m by the way.

3 Applying the Method

By using 2D-TFs, the main approach of this section would be finding the solution of nonlinear two-dimensional mixed Volterra–Fredholm integro-differential equations. Since we show already some of the following statements, one may can conclude and write that:

$$\begin{aligned}
u(s,t) &= U^T T(s,t),\\
f(s,t) &= F^T T(s,t),\\
[u(x,y)]^p &= T^T(x,y) C_p,\\
u_{ss}(s,t) &= U_{ss}^T T(s,t),\\
u_{tt}(s,t) &= U_{tt}^T T(s,t),\\
u_{ts}(s,t) &= U_{ts}^T T(s,t),\\
k(s,t,x,y) &= T^T(s,t)\cdot K \cdot T(x,y),
\end{aligned} \tag{40}$$

where the m_1m_2-vectors $U, G, C_p, U_s, U_t, U_{ss}, U_{tt}, U_{ts}$ and matrix K are the TFs coefficients of $u(s,t)$, $f(s,t)$, $[u(s,y)]^p$, $u_s(s,t)$, $u_t(s,t)$, $u_{ss}(s,t)$, $u_{tt}(s,t)$, u_{st} and $K(s,t,x,y)$ respectively. Now, consider the following equation,

$$u_{ss} + u_{ts} + u_{tt} + u(s,t) = f(s,t) \quad + \quad \int_0^s \int_0^1 k(s,t,x,y)[u(x,y)]^p dydx. \tag{41}$$

The proposed equations and by using (19), (27) and (40), we can conclude that

$$\begin{aligned}
T^T(s,t)(U_{ss}+U_{ts}+U_{tt}+U) &= T^T(s,t)\cdot F + \int_0^s \int_0^1 k(s,t,x,y)[u(x,y)]^p dydx,\\
&\simeq T^T(s,t)\cdot F + \int_0^s \int_0^1 T^T(s,t)\cdot K\cdot T(x,y)\cdot T^T(x,y)\cdot C_p dydx,\\
&= T^T(s,t)\cdot F + T^T(s,t)\cdot K \cdot \int_0^s \int_0^1 T(x,y)\cdot T^T(x,y)\cdot C_p dydx,\\
&\simeq T^T(s,t)\cdot F + T^T(s,t)\cdot K \cdot \int_0^s \int_0^1 R(y)\cdot \tilde{C_p}\cdot Ts(x) dydx,\\
&= T^T(s,t)\cdot F + T^T(s,t)\cdot K \cdot \int_0^1 R(y)dy \cdot \tilde{C_p}\cdot \int_0^s Ts(x)dx,
\end{aligned} \tag{42}$$

where $(4m_1m_2 \times 4m_1m_2)$-matrix $R(y)$ and $4m_1m_2$-vector $Ts(x)$ are determined in Eq. (27). The first integral approximation in Eq. (42) can be revaluated as follows.

$$\int_0^1 R(y)dy \simeq \int_0^1 \begin{bmatrix} \tilde{B_1}(y) & \tilde{B_2}(y) & 0 & 0\\ \tilde{B_2}(y) & \tilde{B_3}(y) & 0 & 0\\ 0 & 0 & \tilde{B_1} & \tilde{B_2}\\ 0 & 0 & \tilde{B_2} & \tilde{B_3} \end{bmatrix} dy, \tag{43}$$

$$\simeq \begin{bmatrix} \frac{h_2}{3}I_1 & \frac{h_2}{6}I_1 & 0 & 0\\ \frac{h_2}{6}I_1 & \frac{h_2}{3}I_1 & 0 & 0\\ 0 & 0 & \frac{h_2}{3}I_1 & \frac{h_2}{6}I_1\\ 0 & 0 & \frac{h_2}{6}I_1 & \frac{h_2}{3}I_1 \end{bmatrix} = \Upsilon,$$

where we put $I_1 = I_{m_1 m_2 \times m_1 m_2}$, for convenience. So,

$$\int_0^s \int_0^1 k(s,t,x,y)[u(x,y)]^p dydx \simeq T^T(s,t) \cdot \left(K \cdot \Upsilon \cdot \tilde{C}_p \cdot E\right) \cdot T(s,t), \tag{44}$$

$$\simeq (\widehat{K\Upsilon\tilde{C}_p E}) \cdot T(s,t), \tag{45}$$

where $(\widehat{K\Upsilon\tilde{C}_p E})$ is a $4m_1m_2$-vector with components which are equal to the diagonal components of the matrix $K\Upsilon\tilde{C}_p E$. Since $\tilde{C}_p$ is a diagonal matrix, we get

$$(\widehat{K\Upsilon\tilde{C}_p E}) = \Pi \cdot C_p, \tag{46}$$

in which Π is a $(4m_1m_2 \times 4m_1m_2)$-matrix with components

$$\Pi_{i,j} = (K\Upsilon)_{i,j} \cdot E_{j,i}, \quad i,j = 1,2,\cdots,4m_1m_2. \tag{47}$$

$$T^T(s,t)(U_{ss} + U_{ts} + U_{tt} + U) = T^T(s,t) \cdot F + T^T(s,t) \cdot \Pi \cdot C_p.$$

Hence, we have

$$U_{ss} + U_{ts} + U_{tt} + U = F + \Pi \cdot C_p. \tag{48}$$

Now, using Equations (31), (32), (35), (36), (39), and (48) we can derive a nonlinear system, also we can use the Newton-Raphson method in order to find the solution of which. Therefore, the approximate solution

$$u(s,t) = U^T T(s,t), \tag{49}$$

can be computed for Eq. (41).

4 Numerical Results

In this section we explain the results obtained with some experiments of nonlinear mixed Volterra-Fredholm integral-differential equations. We use the exact solution in all examples as well as the supplementary initial conditions. The TFs method described in this paper offers a procedure that is practical for finding numerical solutions of examples. The following error function can be used to evaluate both the results of exact solutions and their associated error.

$$e(s,t) = \mid u(s,t) - \bar{u}_{m_1,m_2}(s,t) \mid,$$

where $u(s,t)$ is the exact solution and $\bar{u}_{m_1,m_2}(s,t)$ is the approximate solutions of the integral equation. Tables 1-2 contain the values of $e(s,t)$ with the different values of m_1 and m_2 which is computed as well as displayed over the set

$$D_{grids} = \{(0.0,0.0),(0.1,0.1),(0.2,0.2)\cdots,(0.9,0.9)\}. \tag{50}$$

Example 4.1. *Let the following equation for the first example as:*

$$\frac{\partial^2 u(s,t)}{\partial s^2} + t^3 u(s,t) + \int_0^t \int_0^1 y^2 e^{2z} u^2(y,z)dydz = g(s,t), \quad s,t \in [0,1],$$

where

$$g(s,t) = -\frac{1}{12}t^3 + \frac{1}{12}t^3 e^2 - t^3 e^{s-t} + te^{s-t},$$

where the $u(s,t) = te^{s-t}$ is the exact solution of this problem with supplementary conditions:

$$u(0,t) = te^{-t}, \quad \frac{\partial u}{\partial s}(0,t) = te^{-t}. \tag{51}$$

The results of the numerical solution of this example is displayed in Table 1.

Example 4.2. *For the second example, let the two-dimensional mixed Volterra–Fredholm integro-differential equation as*

$$\frac{\partial^2 u(s,t)}{\partial t^2} + u(s,t) + \int_0^t \int_0^1 s^2 e^{y+z} u(y,z)dydz = g(s,t), \quad s,t \in [0,1],$$

where

$$g(s,t) = -\frac{1}{2}s^2 t + \frac{1}{2}s^2 e^2 t + 2e^{s+t},$$

where $u(s,t) = e^{s+t}$ is the exact solution of this problem and the supplementary conditions are

$$u(s,0) = e^s, \quad \frac{\partial u}{\partial t}(s,0) = e^s,$$

in Table 2, the numerical solutions are presented.

Table 1: Error estimation related to the Example 1.

$s=t$	e(s,t)	e(s,t)	e(s,t)
	$m_1 = m_2 = 4$	$m_1 = m_2 = 8$	$m_1 = m_2 = 32$
0	2.15412×10^{-2}	1.17425×10^{-3}	2.12965×10^{-5}
0.1	1.12941×10^{-2}	1.53289×10^{-3}	1.24598×10^{-5}
0.2	5.24154×10^{-2}	1.36692×10^{-3}	1.26548×10^{-4}
0.3	8.22541×10^{-2}	2.30314×10^{-3}	2.52784×10^{-4}
0.4	7.15222×10^{-2}	8.02847×10^{-3}	8.15252×10^{-4}
0.5	1.25859×10^{-2}	7.01514×10^{-2}	2.23160×10^{-4}
0.6	2.26554×10^{-2}	2.72514×10^{-2}	5.46124×10^{-4}
0.7	2.25541×10^{-2}	1.12342×10^{-2}	7.12464×10^{-4}
0.8	3.27514×10^{-2}	7.06458×10^{-2}	1.02100×10^{-3}
0.9	4.17894×10^{-2}	3.00872×10^{-2}	5.02154×10^{-3}

Table 2: Error estimation related to the Example 2.

$s=t$	e(s,t)	e(s,t)	e(s,t)
	$m_1=m_2=4$	$m_1=m_2=8$	$m_1=m_2=32$
0	2.65214×10^{-2}	5.36982×10^{-3}	2.10963×10^{-5}
0.1	2.36541×10^{-2}	4.25209×10^{-3}	2.60058×10^{-5}
0.2	2.12010×10^{-2}	1.21405×10^{-3}	1.20118×10^{-5}
0.3	5.20514×10^{-2}	1.98740×10^{-3}	2.25124×10^{-4}
0.4	4.95022×10^{-2}	2.25014×10^{-3}	1.02524×10^{-4}
0.5	1.25142×10^{-2}	2.25140×10^{-3}	4.11263×10^{-4}
0.6	3.25478×10^{-2}	1.25105×10^{-2}	1.24502×10^{-4}
0.7	1.21540×10^{-2}	9.14050×10^{-2}	2.41298×10^{-4}
0.8	2.55840×10^{-2}	7.29825×10^{-2}	2.25487×10^{-3}
0.9	2.36542×10^{-2}	1.25005×10^{-2}	5.95015×10^{-3}

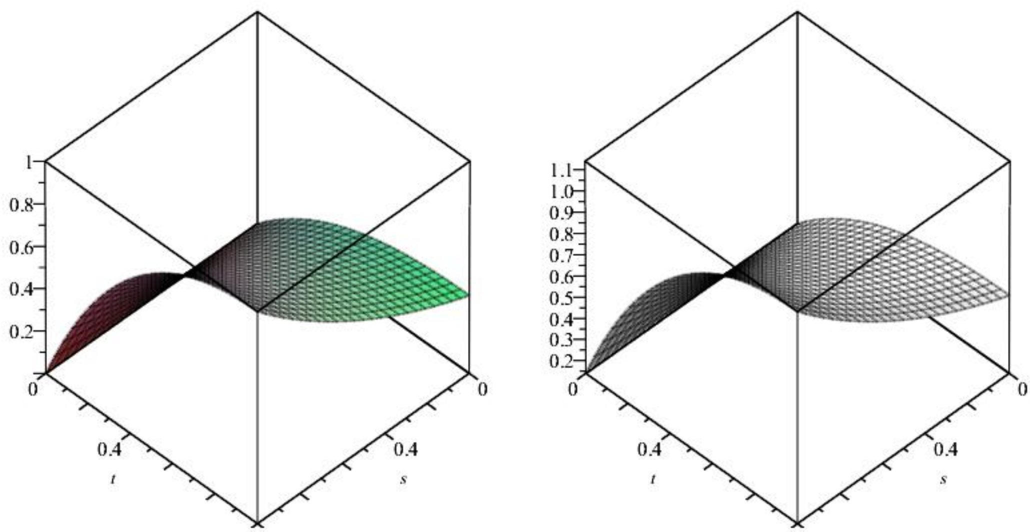

Figure 1: Comparing exact (Right) and numerical (Left) solutions, $u(s,t)$, with $m_1=m_2=32$ for example 1.

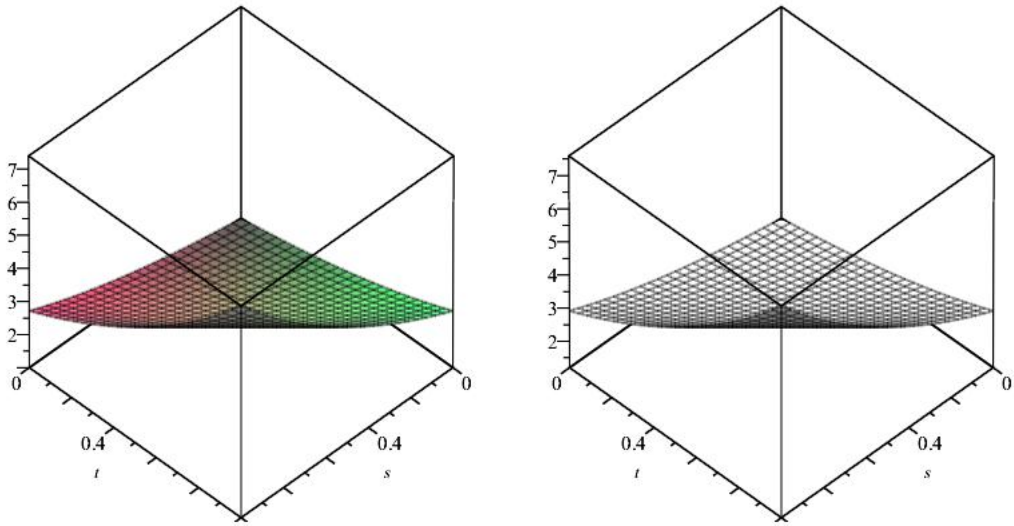

Figure 2: Comparing exact (Right) and numerical (Left) solutions, $u(s,t)$, with $m_1 = m_2 = 32$ for example 2.

5 Conclusion

In this paper, the TFs have been successfully employed to find the approximate solution of the nonlinear mixed Volterra-Fredholm integro-differential equations. This method for constructing a system of algebraic equations without using any projection method like the Collocation, Galerkin, and so on. The biggest advantage of the method is the low cost of computational operations. The accuracy and applicability have been investigated in some examples. The results have been proving the accuracy of the methods which stand at an adequate level of satisfaction. Moreover, to reach an appropriate accuracy, we can increase the rate of m_1 and m_2.

Acknowledgement The authors gratefully acknowledge various helpful comments and suggestions made by the reviewers.

Conflicts of Interest The authors declare no conflict of interest.

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